Braneworld Effective Action at Low Energies

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Abstract. The low energy effective theory for the Randall-Sundrum two-brane system is investigated with an emphasis on the role of the non-linear radion in the brane world. It is shown that the gravity on the brane world is described by a quasi-scalar-tensor theory with a specific coupling function.

Keywords: holographic, brane, effective action

1. Introduction

Motivated by the recent development of the superstring theory, the brane world scenario has been studied intensively. So far, the works are mostly restricted to the linear theory or to homogeneous cosmological models. Here, we consider the nonlinear brane gravity and derive the effective equations of motion for this system using a low energy expansion method developed by us (Kanno and Soda, 2002a; Kanno and Soda, 2002b).

In this paper, we show that the radion disentangles the non-locality in the non-conventional Einstein equations and leads to the local quasi-scalar-tensor gravity.

2. Low Energy Expansion

2.1. RS1 Model and Basic Equations

We consider an S_1/Z_2 orbifold spacetime with the two branes as the fixed points. In the RS1 model (Randall and Sundrum, 1999), the two flat 3-branes are embedded in the 5-dimensional asymptotically anti-deSitter (AdS) bulk with the curvature radius l.

For general non-flat branes, we can not keep both of the two branes straight in the Gaussian normal coordinate system. Hence, we use the following coordinate system to describe the geometry of the brane

model; $ds^2 = e^{2\phi(x^\mu)}dy^2 + g_{\mu\nu}(y,x^\mu)dx^\mu dx^\nu$. We place the branes at $y=0$ (A-brane) and $y=l$ (B-brane) in this coordinate system. The proper distance between the two branes with fixed x coordinates can be written as $d(x) = e^{\phi(x)}l$. Hence, we call ϕ the radion. We introduce the tensor $K_{\mu\nu} = -g_{\mu\nu,y}/2$ for convenience. Decompose the extrinsic curvature into the traceless part and the trace part $e^{-\phi}K_{\mu\nu} = \Sigma_{\mu\nu} + 1/4\ g_{\mu\nu}Q$, $\quad Q = -e^{-\phi}\frac{\partial}{\partial y}\log\sqrt{-g}$. Then, off the brane, we obtain the basic equations;

$$e^{-\phi}\Sigma^\mu{}_{\nu,y} - Q\Sigma^\mu{}_\nu = -\left[\overset{(4)}{R^\mu_{\ \nu}} - \nabla^\mu\nabla_\nu\phi - \nabla^\mu\phi\nabla_\nu\phi\right]_{\rm traceless} , \tag{1}$$

$$\frac{3}{4}Q^2 - \Sigma^\alpha{}_\beta\Sigma^\beta{}_\alpha = \left[\overset{(4)}{R}\right] + \frac{12}{l^2} , \tag{2}$$

$$e^{-\phi}Q_{,y} - \frac{1}{4}Q^2 - \Sigma^{\alpha\beta}\Sigma_{\alpha\beta} = \nabla^\alpha\nabla_\alpha\phi + \nabla^\alpha\phi\nabla_\alpha\phi - \frac{4}{l^2} , \tag{3}$$

$$\nabla_\lambda\Sigma_\mu{}^\lambda - \frac{3}{4}\nabla_\mu Q = 0 , \tag{4}$$

where the subscript "traceless" represents the traceless part of the quantity in the square brackets and $R^\mu{}_\nu{}^{(4)}$ is the curvature on the brane and ∇_μ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$. And the junction conditions are

$$\left[\Sigma^\mu_\nu - \frac{3}{4}\delta^\mu_\nu Q\right]\Big|_{y=0} = \frac{\kappa^2}{2}(-\sigma_A\delta^\mu_\nu + T^{A\mu}{}_\nu) , \tag{5}$$

$$\left[\Sigma^\mu_\nu - \frac{3}{4}\delta^\mu_\nu Q\right]\Big|_{y=l} = -\frac{\kappa^2}{2}(-\sigma_B\delta^\mu_\nu + \tilde{T}^{B\mu}{}_\nu) . \tag{6}$$

2.2. Low Energy Expansion Scheme

In this paper, we will consider the low energy regime in the sense that the energy density of the matter, ρ_i (i=A,B), on a brane is much smaller than the brane tension, i.e., $\rho_i/|\sigma_i| \ll 1$. In this regime, a simple dimensional analysis, $\rho_i/|\sigma_i| \sim \frac{l}{\kappa^2|\sigma_i|}\frac{\kappa^2\rho_i}{l} \sim (l/L)^2 \ll 1$ implies that the curvature on the brane can be neglected compared with the extrinsic curvature at low energies.

Our iteration scheme is to write the metric $g_{\mu\nu}$ as a sum of local tensors built out of the induced metric on the brane, with the number of derivatives increasing with the order of iteration, that is, $O((l/L)^{2n})$, $n = 0,1,2,\cdots$. Hence, we seek the metric as a perturbative series

$$g_{\mu\nu}(y,x^\mu) = a^2(y,x)\left[h_{\mu\nu}(x^\mu) + g^{(1)}_{\mu\nu}(y,x^\mu) + g^{(2)}_{\mu\nu}(y,x^\mu) + \cdots\right] , \tag{7}$$

$$g^{(n)}_{\mu\nu}(y=0,x^\mu)=0\ , \quad n=1,2,3,... \tag{8}$$

where the factor $a^2(y,x)$ is extracted because of the reason explained later and we put the Dirichlet boundary condition $g_{\mu\nu}(y=0,x)=h_{\mu\nu}(x)$ at the A-brane.

3. Background Geometry

As we can ignore the matter at the lowest order, we obtain the vacuum brane. Namely, we have an almost flat brane compared with the curvature scale of the bulk space-time. The term $\Sigma^{(0)\mu}{}_\nu$ is not allowed to exist because of the junction conditions (5) and (6). Then, it is easy to solve the remaining equations. The result is $\Sigma^{(0)\mu}{}_\nu=0,\ Q^{(0)}=4/l$. Using the definition $K^{(0)}_{\mu\nu}=-\frac{1}{2}\frac{\partial}{\partial y}g^{(0)}_{\mu\nu}=\frac{1}{l}e^\phi g^{(0)}_{\mu\nu}$, we get the 0-th order metric as

$$ds^2=e^{2\phi(x)}dy^2+a^2(y,x)h_{\mu\nu}(x^\mu)dx^\mu dx^\nu, \quad a(y,x)=\exp\left[-e^{\phi(x)y/l}\right], \tag{9}$$

where the tensor $h_{\mu\nu}$ is the induced metric on the A-brane.

Given the 0-th order solution, junction conditions (5) and (6) lead to the well known relations $\kappa^2\sigma_A=6/l\ ,\kappa^2\sigma_B=-6/l$. Note that $\phi(x)$ and $h_{\mu\nu}(x)$ are arbitrary functions of x at the 0-th order.

4. Holographic Gravity

4.1. Bulk Geometry

The next order solution is obtained by taking into account the terms neglected at the 0-th order. It is at this order that the effect of matter comes in. We obtain the trace part of the extrinsic curvature as

$$Q^{(1)}=\frac{l}{a^2}\left[\frac{1}{6}R(h)+\frac{ye^\phi}{l}\left(\phi^{|\alpha}{}_{|\alpha}+\phi^{|\alpha}\phi_{|\alpha}\right)-\frac{y^2e^{2\phi}}{l^2}\phi^{|\alpha}\phi_{|\alpha}\right], \tag{10}$$

where the superscript (1) represents the order of the gradient expansion. It is convenient to introduce the Ricci tensor of $h_{\mu\nu}$, denoted by $R^\mu_\nu(h)$. Hereafter, we omit the argument of the curvature for simplicity. The traceless part of the extrinsic curvature is

$$\begin{aligned}\Sigma^{(1)\mu}{}_\nu \ =\ & \frac{l}{a^2}\left[\frac{1}{2}\left(R^\mu{}_\nu-\frac{1}{4}\delta^\mu_\nu R\right)+\frac{ye^\phi}{l}\left(\phi^{|\mu}{}_{|\nu}-\frac{1}{4}\delta^\mu_\nu\phi^{|\alpha}{}_{|\alpha}\right)\right.\\ & \left.+\left(\frac{y^2e^{2\phi}}{l^2}+\frac{ye^\phi}{l}\right)\left(\phi^{|\mu}\phi_{|\nu}-\frac{1}{4}\delta^\mu_\nu\phi^{|\alpha}\phi_{|\alpha}\right)\right]+\frac{\chi^\mu{}_\nu(x)}{a^4},\end{aligned} \tag{11}$$

where χ^μ_ν is an integration constant with the property $\chi^\mu{}_\mu = 0$. And χ^μ_ν must be transverse $\chi^\mu{}_{\nu|\mu} = 0$ in order to satisfy Eq. (4). From these results, we can obtain the bulk metric:

$$
\begin{aligned}
g^{(1)}_{\mu\nu} \;=\; & -\frac{l^2}{2}\left(\frac{1}{a^2}-1\right)(R_{\mu\nu}-\frac{1}{6}h_{\mu\nu}R)+\frac{l^2}{2}\left(\frac{1}{a^2}-1-\frac{2ye^\phi}{l}\frac{1}{a^2}\right)\times \\
& \left(\phi_{|\mu\nu}+\frac{1}{2}h_{\mu\nu}\phi^{|\alpha}\phi_{|\alpha}\right)-\frac{y^2e^{2\phi}}{a^2}\left(\phi_{|\mu}\phi_{|\nu}-\frac{1}{2}h_{\mu\nu}\phi^{|\alpha}\phi_{|\alpha}\right) \\
& -\frac{l}{2}\left(\frac{1}{a^4}-1\right)\chi_{\mu\nu}\ , \qquad (12)
\end{aligned}
$$

The term of $\chi_{\mu\nu}$ is essentially the Weyl tensor at this order. Note that we have obtained the bulk metric in terms of $\phi(x)$, $h_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$.

4.2. Quasi-Scalar-Tensor Gravity

We shall deduce the equations for $\phi(x)$, $h_{\mu\nu}(x)$ and $\chi_{\mu\nu}(x)$ from junction conditions. The junction condition at the A-brane is written as

$$
\frac{l}{2}G^\mu{}_\nu(h)+\chi^\mu{}_\nu=\frac{\kappa^2}{2}T^{A\mu}{}_\nu\ . \qquad (13)
$$

This equation is nothing but the Einstein equations with the generalized dark radiation $\chi_{\mu\nu}$. It should be noted that $\chi_{\mu\nu}$ is undetermined at this level, exhibiting the non-local nature of Eq. (13).

The junction condition at the B-brane is given by

$$
\frac{l}{2}G^\mu{}_\nu(f)+\frac{\chi^\mu{}_\nu}{\Omega^4}=-\frac{\kappa^2}{2}T^{B\mu}{}_\nu\ . \qquad (14)
$$

where $f_{\mu\nu}$ is the induced metric on the B-brane and $\Omega(x) = a(y = l, x) = \exp[-e^\phi]$. Here, the index of $T^{B\mu}{}_\nu$ is the energy momentum tensor with the index raised by the induced metric on the B-brane. It should be noted that $h_{\mu\nu}$ and $f_{\mu\nu}$ are not independent, but it is related as $f_{\mu\nu} = \Omega^2 h_{\mu\nu}$ at this order. Although Eqs. (13) and (14) are non-local individually, with undetermined $\chi_{\mu\nu}$, one can combine both equations to reduce them to local equations for each brane. This happens to be possible since $\chi_{\mu\nu}$ appears only algebraically; one can easily eliminate $\chi_{\mu\nu}$ from Eqs. (13) and (14). Defining a new field $\Psi = 1 - \Omega^2$, we find

$$
\begin{aligned}
G^\mu{}_\nu(h) \;=\; & \frac{\kappa^2}{l\Psi}T^{A\mu}{}_\nu+\frac{\kappa^2(1-\Psi)^2}{l\Psi}T^{B\mu}{}_\nu \\
+\; & \frac{1}{\Psi}\left(\Psi^{|\mu}{}_{|\nu}-\delta^\mu_\nu\Psi^{|\alpha}{}_{|\alpha}\right)+\frac{\omega(\Psi)}{\Psi^2}\left(\Psi^{|\mu}\Psi_{|\nu}-\frac{1}{2}\delta^\mu_\nu\Psi^{|\alpha}\Psi_{|\alpha}\right), \qquad (15)
\end{aligned}
$$

where the coupling function $\omega(\Psi)$ takes the form: $\omega(\Psi) = 3\Psi/2(1-\Psi)$. We can also determine χ^μ_ν by eliminating G^μ_ν from Eqs. (13) and (14). Then,

$$\begin{aligned}\chi^{\mu}{}_{\nu} \; = \; & -\frac{\kappa^2(1-\Psi)}{2\Psi}\left(T^{A\mu}{}_{\nu} + (1-\Psi)T^{B\mu}{}_{\nu}\right) \\ & -\frac{l}{2\Psi}\left[\left(\Psi^{|\mu}{}_{|\nu} - \delta^\mu_\nu \Psi^{|\alpha}{}_{|\alpha}\right) + \frac{\omega(\Psi)}{\Psi}\left(\Psi^{|\mu}\Psi_{|\nu} - \frac{1}{2}\delta^\mu_\nu \Psi^{|\alpha}\Psi_{|\alpha}\right)\right] \quad (16)\end{aligned}$$

The condition $\chi^{\mu}{}_{\mu} = 0$ gives rise to the field equation for Ψ:

$$\Box\Psi = \frac{\kappa^2}{l}\frac{T^A + (1-\Psi)T^B}{2\omega+3} - \frac{1}{2\omega+3}\frac{d\omega}{d\Psi}\Psi^{|\mu}\Psi_{|\mu} \ , \qquad (17)$$

where we have used the explicit form of $\omega(\Psi)$. This equation tells us that the trace part of the energy momentum tensor determines the radion field and hence the relative bending of the brane. And $\chi_{\mu\nu}$ is determined by the traceless part of the right-hand side of Eq. (16). Remarkably, $\chi_{\mu\nu}$ is now a secondary entity.

The action is derived from the original 5-dimensional action by substituting the solution of the equation of motion in the bulk. Up to the first order , we obtain

$$\begin{aligned}S_A \; = \; & \frac{l}{2\kappa^2}\int d^4x\sqrt{-h}\left[\Psi R(h) - \frac{\omega(\Psi)}{\Psi}\Psi^{|\alpha}\Psi_{|\alpha}\right] \\ & + \int d^4x\sqrt{-h}\mathcal{L}^A + \int d^4x\sqrt{-h}\,(1-\Psi)^2\,\mathcal{L}^B \ . \qquad (18)\end{aligned}$$

Eqs. (15) and (17) are the basic equations to be used in cosmological or astrophysical contexts when the characteristic energy density is less than $|\sigma_i|$. In contrast to the usual scalar-tensor gravity, this theory couples with two kinds of matter, namely, the matters on both positive and negative tension branes, with different effective gravitational coupling constants. For this reason, we call this theory the quasi-scalar-tensor gravity. Thus, the (non-local) Einstein equations (13) with the generalized dark radiation has transformed into the (local) quasi-scalar-tensor gravity (15) with the coupling function $\omega(\Psi)$.

Here, it should be noted that χ^μ_ν which appeared in $g^{(1)}_{\mu\nu}$ is a non-local quantity. In fact, eliminating Ψ from Eq. (16) by solving Eq. (17) yields a non-local expression of χ^μ_ν. If we substitute this non-local expression into Eq. (13), we will obtain a non-local theory. Conversely, one can see that introducing the radion disentangles the non-locality in the non-conventional Einstein equations (13) and yields the quasi-scalar-tensor gravity given by Eqs. (15) and (17).

For completeness, we shall derive the effective action on the B-brane. Substituting $h_{\mu\nu} = \Omega^{-2} f_{\mu\nu}$ into Eq. (18) and defining $\Phi = \Omega^{-2} - 1$, we obtain the effective action on the B-brane

$$
\begin{aligned}
S_B \;=\; & \frac{l}{2\kappa^2} \int d^4x \sqrt{-f} \left[\Phi R(f) - \frac{\omega(\Phi)}{\Phi} \Phi^{;\alpha} \Phi_{;\alpha} \right] \\
& + \int d^4x \sqrt{-f} \mathcal{L}^B + \int d^4x \sqrt{-f} \mathcal{L}^A (1+\Phi)^2 . \qquad (19)
\end{aligned}
$$

where the coupling function $\omega(\Phi)$ takes the form: $\omega(\Phi) = -3\Psi/2(1+\Phi)$ and ; denote the covariant derivative with respect to the metric $f_{\mu\nu}$.

5. Conclusion

We have developed a method to deduce the low energy effective theory for the two-brane system. As a result, we have shown that the gravity on the brane world is described by a quasi-scalar-tensor theory with a specific coupling function $\omega(\Psi) = 3\Psi/2(1-\Psi)$ on the positive tension brane and $\omega(\Phi) = -3\Phi/2(1+\Phi)$ on the negative tension brane, where Ψ and Φ are Brans-Dicke-like scalars on the positive and negative tension branes, respectively.

In the process of derivation of the effective equations of motion, we have clarified how the quasi-scalar tensor gravity emerges from Einstein's theory with the generalized dark radiation term described by $\chi_{\mu\nu}$.

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